Rev. Ivoir. Sci. Technol., 23 (2014) 54 - 71
ISSN 1813-3290, http://www.revist.ci

INTERFACE CRACK LOADED IN TENSION: INTRODUCING AN INTERNAL SHEAR STRESS PROMOTED BY THE POISSON'S EFFECT

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ABSTRACT

This study takes into account the existence of Poisson's effect when an interface crack of finite length $2a$, in the Ox_1x_3-plane, is loaded in tension σ_{22}^a along the x_2- direction of a Cartesian coordinate system x_i. The resultant internal shear stress τ_{12}^a is identified, estimated in the plane of the interface, and introduced in the conditions for the propagation of the crack. In the framework of linear elasticity, τ_{12}^a is assumed to be proportional to position along the interface. The crack is represented by a continuous distribution of two families of edge dislocations with infinitesimal Burgers vectors. The distribution functions D_1 and D_2 of the dislocations at equilibrium satisfy a system of two integral equations with Cauchy-type singular kernels.

A solution is given to a class of singular integral equations which, when applied to our modelling, permits to derive closed-form expressions for various physical quantities pertinent to discuss crack motion, namely, the dislocation distribution functions D_1 and D_2 with corresponding relative displacements ϕ_1 and ϕ_2 of the faces of the crack, the crack-tip stresses $\bar{\sigma}_{12}(s)$ and $\bar{\sigma}_{22}(s)$, and the crack extension force G. When neglecting Poisson effect, complete agreement with previous studies on the interface crack is achieved. Poisson's effect gives higher values to the crack extension force.

Keywords : *linear elasticity, poisson effect, dislocations, interface crack, singular integral equations, brittle fracture mechanics.*

RÉSUMÉ

Fissure d'interface sollicitée en tension: présentation d'une contrainte interne de cisaillement générée par l'effet Poisson

La présente étude prend en compte l'existence de l'effet Poisson au cours de la sollicitation en tension d'une fissure d'interface. La contrainte interne de cisaillement qui en résulte est identifiée, estimée dans le plan de l'interface et introduite dans l'analyse des conditions de propagation de la fissure. Par rapport à un système d'axes cartésiens x_i, la fissure de longueur finie $2a$ est située à l'origine dans le plan Ox_1x_3 avec un front droit parallèle à x_3 et sollicitée en tension σ_{22}^a dans la direction x_2. Dans le cadre de l'élasticité linéaire, nous proposons une contrainte interne induite τ_{12}^a proportionnelle à la position le long de l'interface. La fissure est représentée par une distribution continue de deux familles de dislocations coins de vecteurs de Burgers infinitésimaux.

Les fonctions de distribution D_1 et D_2 des dislocations à l'équilibre satisfont un système de deux équations intégrales avec des singularités de type Cauchy. Une solution est proposée pour une certaine classe d'équations intégrales singulières, laquelle appliquée à notre modèle, fournit sous expressions mathématiques fermées diverses grandeurs physiques pertinentes pour discuter des conditions de propagation des fissures, notamment, les fonctions de distribution des dislocations de fissure D_1 et D_2 et les déplacements relatifs correspondant ϕ_1 et ϕ_2 des lèvres de la fissure, les contraintes en tête de fissure $\bar{\sigma}_{12}(s)$ et $\bar{\sigma}_{22}(s)$, et la force d'extension G de la fissure. Lorsque l'effet Poisson est négligé, nos résultats sont en parfait accord avec des études antérieures. L'effet Poisson fournit des valeurs plus élevées de G.

Mots-clés : *élasticité linéaire, effet Poisson, dislocations, fissure d'interface, équations intégrales singulières, mécanique de la rupture fragile.*

I - INTRODUCTION

Consider two solids (1) and (2) ***(Figure 1)***, isotropic and elastic, with notation ν_i, μ_i, E_i ($i = 1$ and 2) to designate Poisson's ratio, shear modulus and Young's modulus respectively. We assume that the two solids are firmly linked along a planar interface, such that: mediums 1 and 2 occupy the regions $x_2 > 0$ and $x_2 < 0$ respectively; Ox_1x_3 represents the interface with origin O at the centre. When the system is subjected to a uniform applied tension σ_{22}^a at infinity along the x_2 – direction, the two media suffer, far from the interface, a uniform contraction $(-\nu_i\sigma_{22}^a / E_i)$ ($i = 1$ and 2) in the x_1 and x_3 directions; this is known as Poisson's effect whose physical origin is associated to the existence of "transverse" microscopic bonds. The stress field is modified by the presence of the interface so that the surrounding medium suffers at an

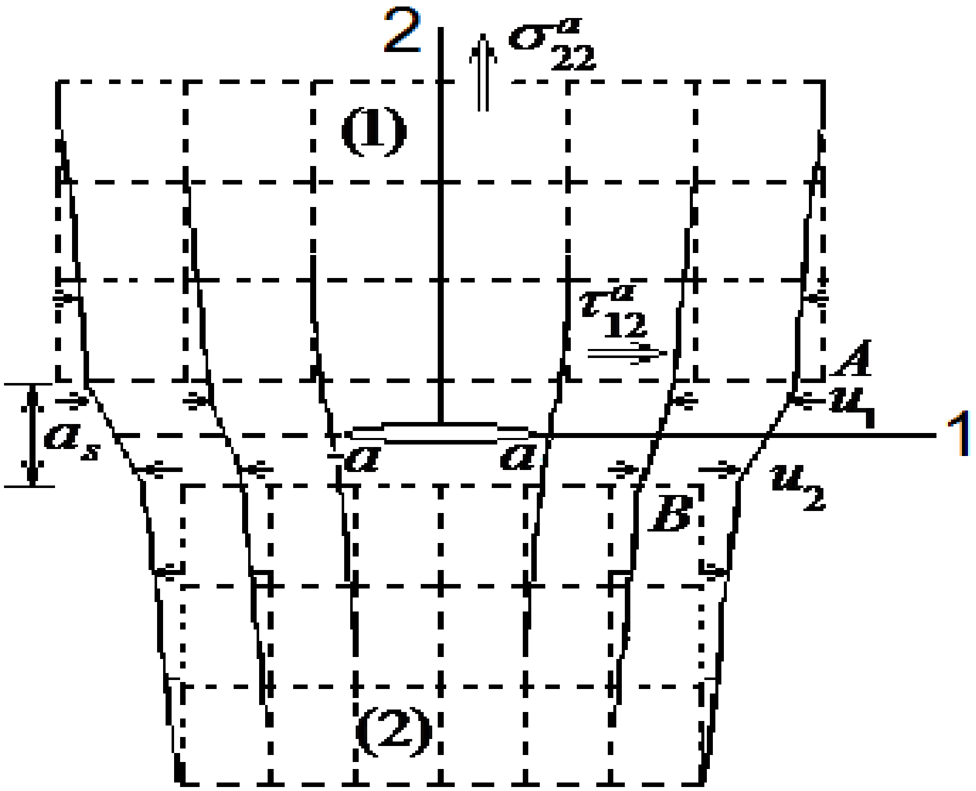

Figure 1 : *Two elastic media (1) and (2) subjected to a uniform tension σ_{22}^a at infinity. Far from the interface (Ox_1x_3), the media suffered uniform Poisson's contractions $(-\nu_i\sigma_{22}^a / E_i)$ ($i = 1$ and 2) in the x_1 – direction. The induced shear stress τ_{12}^a is indicated about the interface. A crack of finite length 2a located at the origin is present. This sketch corresponds to $(\nu_2\sigma_{22}^a / E_2) > (\nu_1\sigma_{22}^a / E_1)$. See text for additional comments.*

Arbitrary position *P*, with coordinates (x_1, x_2, x_3), internal shear stresses (see Section 2.1) $\tau_{12}^a(P)$ and $\tau_{23}^a(P)$ due to the difference in the values of the Poisson's contraction in the media 1 and 2 when crossing the interface. These shear stresses at *P*, for fixed x_1 and x_3, increase in magnitude as one approaches the interface ($x_2 \to 0$) and decrease to zero far away from the interface ($x_2 \to \pm\infty$). In the present work, we provide an expression to τ_{12}^a on the plane of the interface and then we introduce this value in an analysis of an interface crack loaded in tension. The considered crack has a finite length $2a$ and is located in the Ox_1x_3-plane, ***Figure 1***; it extends from $x_1 = -a$ to a with a straight front running indefinitely in the x_3-direction.

This study is basic because Poisson's effect is observed in almost all real loaded materials; its contribution to the conditions for the propagation of a crack must be evaluated in a number of experimental situations. A similar study has been done recently for the propagation of a brittle crack under compression in an homogeneous solid (Anongba et al. [1]); it is shown there that Poisson's effect promotes the axial failure of the specimen. Because we are facing a crack problem in the plane theory of elasticity, a first general solution procedure can be to express the stresses and strains in terms of the Airy stress function and then introduce the complex representation; the elastic fields are real and imaginary parts of complex functions that satisfy the boundary conditions. This method, expounded by Muskhelishvili [2], allows a simplified mathematical treatment.

This procedure (with some variants) is effective for the determination of stresses and relative displacement of the faces of the crack at the very tip of the crack, particularly when a Taylor series expansion is made of the complex analytical functions: see Erdogan [3] , Rice and Sih [4] and Rice [5]; additional references are given by Rice [5]. Another way to deal with cracks is to represent them by a continuous distribution of dislocation families with infinitesimal Burgers vectors. The stress field induced by the crack in the surrounding medium will be given by the dislocations of the distribution. Bilby and Eshelby [6] have given a basic treatment of the method where they have treated the case of planar arrays of straight dislocations in an isotropic elastic medium. We have used this technique extensively and also extended the analysis to non-planar cracks by introducing sinusoidal edge and screw dislocations (see Anongba [1, 7 to 12], Anongba and Vitek [13]). This method has also been used for the interface crack with contact zones (see Comninou [14, 15] and Comninou and Dundurs [16], among others).

The first step is to determine the distribution functions of the crack dislocation families at equilibrium under load. We generally arrive at a system of singular integral equations involving these functions. When the distribution functions of the dislocations have been found, we can obtain by integration the relative displacements of the faces of the crack, the crack-tip stresses and the crack extension force. In the present study, assuming the crack faces to be traction free, we shall solve the problem of the interface crack loaded in tension (taking account of lateral Poisson's contractions) by representing the crack by a continuous distribution of straight edge dislocations with infinitesimal Burgers vectors.

We ignore any relative displacement of the faces of the crack in the x_3 – direction (i.e. no screw dislocation family is considered). The first task is to give an expression to the induced shear stress originating from the difference in Poisson's contraction when crossing the interface; this is done in section 2.1. In Section 2.2, our crack model is detailed and the singular integral equations, fulfilled by the distribution functions of the crack dislocations, are given. We obtain a solution to the integral equations in Section 3.1. The various physical quantities required in the analysis of the conditions for the propagation of the interface crack are expressed in Section 3.2. In Sections 4 and 5, a discussion and conclusion are made of our results, respectively.

II - MODELLING METHODOLOGY

II-1. Internal shear stress on the interface

For illustration purpose only of the displacement, we construct on the continuum on both sides of the interface, an identical cubic lattice. We stress immediately that this is not a lattice in the sense of crystallography (i.e. this is not associated with atomic arrangement). We assume that Hooke's law is valid throughout. Under a uniform applied tension σ_{22}^a, the lattices 1 and 2 shrink differently. These are illustrated in dashed lines in ***Figure 1*** under the assumption $\nu_2\sigma_{22}^a / E_2 > \nu_1\sigma_{22}^a / E_1$. Consider two nodes A (medium 1) and B (medium 2) on either side of the interface which have the equal coordinate x_1 (before loading) and a separation distance d (along x_1) under load when the two materials are not welded. In the welded state, A is moved horizontally by a distance of $u_1 < 0$ and B by $u_2 > 0$.

We ignore vertical displacement along x_2. We have:

- $u_1(x_1) < 0$ for $x_1 > 0$ and $u_1(x_1) > 0$ for $x_1 < 0$,
- $u_2(x_1) > 0$ for $x_1 > 0$ and $u_2(x_1) < 0$ for $x_1 < 0$.

u_1 and u_2 are odd functions (we can restrict ourselves to $x_1 > 0$). Let ϕ denotes the relative position of A with respect to B along x_1, then

$$\phi = x_1(A) - x_1(B) = d + u_1(x_1) - u_2(x_1) \quad (1)$$

where

$$d = x_1\left(\frac{\nu_2}{E_2} - \frac{\nu_1}{E_1}\right)\sigma_{22}^a \quad (2)$$

In ***Figure 1***, the displacements u_1 and u_2 at different places are represented by short horizontal arrows and the resultant welded lattices are drawn in solid lines. The distance d (2) increases linearly with the position x_1 along the interface. We can also see from ***Figure 1*** that the magnitude of $(u_1 - u_2)$ also increases with x_1, apparently in proportion to d. We may write

$$\phi = \nu_s d = \nu_s x_1\left(\frac{\nu_2}{E_2} - \frac{\nu_1}{E_1}\right)\sigma_{22}^a \quad (3)$$

where ν_s is a quantity that may actually be x_1-dependent; we shall assume ν_s to be constant in the present study. The validity of this assumption is stressed in the discussion (Section 4). The shear stress τ_{12}^a at point $P(x_1, 0, x_3)$ then takes the form

$$\tau_{12}^a(P) = \mu_s \frac{\phi}{a_s} = -\frac{\nu_s \mu_s}{a_s}\left(\frac{\nu_1}{E_1} - \frac{\nu_2}{E_2}\right)\sigma_{22}^a x_1 \equiv -\bar{\alpha} x_1 \quad (4)$$

where μ_s corresponds to the shear modulus about the interface and a_s is the distance between A and B along x_2 (***Figure 1***). This is that final form which is given to the interfacial internal shear stress. Expression (4) will be used in what follows.

II-2. Model of interfacial crack and singular integral equations

The crack in ***Figure 1*** is assumed to be filled with two families (1 and 2) of straight edge dislocations parallel to x_3 with Burgers vectors $\vec{b}_1 = (b, 0, 0)$ and $\vec{b}_2 = (0, b, 0)$ in the x_1 and x_2 directions, respectively. The system is subjected to uniform σ_{22}^a along x_2 at infinity; in addition we consider internal shear stresses transmitted by the interface as the result of the existence of internal uniform Poisson stresses $(-\nu_1\sigma_{22}^a)$ and $(-\nu_2\sigma_{22}^a)$ in the x_1- direction in materials 1 and 2, respectively. The dislocation distribution function $D_i(x_1)$ ($i=1$ and 2) gives the number of dislocations of family i in a small interval dx_1 about x_1 as $D_i(x_1)dx_1$. To find the equilibrium dislocation distributions, we may ask for zero total force on the crack faces; on our modelling, this gives

$$\begin{cases}\bar{\sigma}_{12}=0\\ \bar{\sigma}_{22}=0\end{cases}; \qquad (5)$$

$\bar{\sigma}_{ij}$ stands for the total stress at any point $P(x_1,x_2,x_3)$ in the surrounding medium and is linked to D_i; in (5), we are only concerned with the points of the crack faces. $\bar{\sigma}_{ij}(P)$ is written as

$$\bar{\sigma}_{ij}=\sigma_{ij}^a+\sigma_{ij}^{(s)}+\bar{\sigma}_{ij}^{(1)}+\bar{\sigma}_{ij}^{(2)} \qquad (6)$$

where σ_{ij}^a corresponds to both the applied tension and the assumed uniform internal Poisson stress, $\sigma_{ij}^{(s)}$ to the stress induced by the interface under load in absence of the crack and

$$\bar{\sigma}_{ij}^{(n)}=\int_{-a}^{a}\sigma_{ij}^{(n)}(x_1-x_1^{'},x_2,x_3)D_n(x_1^{'})dx_1^{'} \ (n=1 \text{ and } 2); \qquad (7)$$

here $\sigma_{ij}^{(n)}$ is the stress field produced by a dislocation of family n at the origin.

In order to write explicitly (5), we display the following results on the interface (points $P(x_1,x_2=0,x_3)$):

- σ_{12}^a is zero except σ_{22}^a;
- $\sigma_{12}^{(s)}$ is given by τ_{12}^a (4) and $\sigma_{22}^{(s)}=0$;
- Edges with $\vec{b}_1=(b,0,0)$ (Family 1)

$$\sigma_{12}^{(1)}(x_1-x_1^{'},x_2=0,x_3)=\frac{bC}{\pi}\frac{1}{x_1-x_1^{'}}$$

$$\sigma_{22}^{(1)}(x_1-x_1^{'},x_2=0,x_3)=-b\beta C\delta(x_1-x_1^{'}); \qquad (8)$$

- Edges with $\vec{b}_2=(0,b,0)$ (Family 2)

$$\sigma_{12}^{(2)}(x_1-x_1^{'},x_2=0,x_3)=b\beta C\delta(x_1-x_1^{'}),$$

$$\sigma_{22}^{(2)}(x_1-x_1^{'},x_2=0,x_3)=\frac{bC}{\pi}\frac{1}{x_1-x_1^{'}}. \qquad (9)$$

In (8) and (9), taken from Comninou and Dundurs [16], δ is the Dirac delta function. Moreover,

$$C = \frac{2\mu_1(1-\alpha)}{(\kappa_1+1)(1-\beta^2)} = \frac{2\mu_2(1+\alpha)}{(\kappa_2+1)(1-\beta^2)} \tag{10}$$

with

$$\alpha = \frac{\mu_1(\kappa_2+1)-\mu_2(\kappa_1+1)}{\mu_1(\kappa_2+1)+\mu_2(\kappa_1+1)}\,;\; -1 \le \alpha \le 1 \tag{11}$$

$$\beta = \frac{\mu_1(\kappa_2-1)-\mu_2(\kappa_1-1)}{\mu_1(\kappa_2+1)+\mu_2(\kappa_1+1)}\,;-\frac{1}{2} \le \beta \le \frac{1}{2} \tag{12}$$

where $\kappa_i = 3-4\nu_i$. The traction free boundary condition (5) yields

$$-\bar{\alpha}x_1\delta_{i1} + \sigma_{22}^a\delta_{i2} + b\beta C\big(D_2(x_1)\delta_{i1} - D_1(x_1)\delta_{i2}\big) + \frac{bC}{\pi}\int_{-a}^{a}\frac{D_i(x_1^{'})}{x_1 - x_1^{'}}dx_1^{'} = 0\,, \tag{13}$$

$i = 1$ and $2, |x_1| < a$, and δ_{ij} is the Kronecker delta. We arrive at a system of two integral equations with Cauchy-type singular kernels with unknown functions D_1 and D_2. The Cauchy principal values of the integrals have to be taken. Next, an analytical solution is given to the governing equations (13).

III - CALCULATION RESULTS

III-1. Analytical solution to singular integral equations

From the second of (13) and using the book by Muskhelishvili [17](see chapter 11 "Inversion formulae for arcs"), we write

$$D_2(x_1) = \frac{1}{\pi bC}\int_{-a}^{a}\sqrt{\frac{a^2 - x_1^{'2}}{a^2 - x_1^2}}\,\frac{\left(\sigma_{22}^a - b\beta C D_1(x_1^{'})\right)}{x_1 - x_1^{'}}dx_1^{'} \tag{14}$$

Introducing (14) in the first equation of (13), we obtain

$$\int_{-a}^{a}\left(\sqrt{a^2-x_1^2}-\beta^2\sqrt{a^2-x_1^{'2}}\right)\frac{D_1(x_1^{'})}{x_1^{'}-x_1}dx_1^{'}=\frac{\pi}{bC}f(x_1) \qquad (15)$$

where

$$f(x_1)=-\bar{\alpha}x_1\sqrt{a^2-x_1^2}+\beta\sigma_{22}^{a}x_1\,. \qquad (16)$$

We temporary modify our notation and write (15) as

$$\frac{\sqrt{1-s^2}}{\pi}\int_{-1}^{1}\frac{g(t)}{t-s}dt+\frac{\lambda^2}{\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}\,g(t)}{t-s}dt=\frac{1}{abC}\bar{f}(s) \qquad (17)$$

Where

$$\begin{aligned} &g(t)=D_1(at)\\ &\bar{f}(s)=f(as) \qquad (18)\\ &\lambda=i\beta \end{aligned}$$

(i. e. $\lambda^2=-\beta^2$). Equation (17) takes the form of "Example 43" (page 58) in the book by Estrada and Kanwal [18]; by a convincing operational approach , it is shown there that the solution of (17) can be written as

$$g(s)=\frac{\bar{\psi}_1(s)-\bar{\psi}_2(s)}{2\lambda\sqrt{1-s^2}} \qquad (19)$$

where $\bar{\psi}_1$ and $\bar{\psi}_2$ are solutions of the pair of Cauchy type integral equations

$$\bar{\psi}_n(s)+(\delta_{n1}-\delta_{n2})\frac{\lambda}{\pi}\int_{-1}^{1}\frac{\bar{\psi}_n(t)}{t-s}dt=\frac{\bar{f}(s)}{abC}+A\delta_{n1}+B\delta_{n2}\,;\ n=1\text{and }2 \qquad (20)$$

that satisfy the additional requirement

$$\frac{1}{\pi}\int_{-1}^{1}g(t)\ln\left(\frac{1+t}{1-t}\right)dt=\int_{-1}^{1}\frac{\bar{f}(t)}{abC\sqrt{1-t^2}}dt\,; \qquad (21)$$

A and B are arbitrary constants.

The solutions of (20) ($\bar{\psi}_1$ and $\bar{\psi}_2$) can be worked out by using the treatment of Muskhelishvili [17] (see chapter 14 "The case of continuous coefficients"); we have

$$\bar{\psi}_n(s) = \frac{1}{1-\beta^2}\left(\frac{\bar{f}(s)}{abC} + A\delta_{n1} + B\delta_{n2}\right) + (\delta_{n1} - \delta_{n2})\frac{1}{\pi i}\left(\frac{\beta}{1-\beta^2}\right)\left(\frac{1-s}{1+s}\right)^{i\delta(-\delta_{n1}+\delta_{n2})}$$

$$\times\left(const_1\delta_{n1} + const_2\delta_{n2} + \int_{-1}^{1}\frac{\left(\bar{f}(t)/abC + A\delta_{n1} + B\delta_{n2}\right)dt}{\left(1-t/1+t\right)^{i\delta(-\delta_{n1}+\delta_{n2})}(t-s)}\right) \qquad (22)$$

Where

$$\delta = \frac{1}{2\pi}\ln\left(\frac{1+\beta}{1-\beta}\right); \qquad (23)$$

$n = 1$ and 2; $const_1$ and $const_2$ are arbitrary constants. Using (21) and some considerations on the nature (odd and even) of $\bar{f}$ and g, we find that $const_1 = const_2$, $A = -B$. Then, a simple form for g follows from which is deduced the following expression for D_1 the distribution function of the edge dislocation family 1:

$$D_1(x_1) = \frac{const}{\pi\sqrt{a^2 - x_1^2}}\mathrm{Re}\left[\left(\frac{a-x_1}{a+x_1}\right)^{i\delta}\right]$$
$$-\frac{1}{\pi bC(1-\beta^2)\sqrt{a^2-x_1^2}}\mathrm{Re}\left[\left(\frac{a-x_1}{a+x_1}\right)^{i\delta}\int_{-a}^{a}\left(\frac{a+t}{a-t}\right)^{i\delta}\frac{f(t)}{t-x_1}dt\right], \quad (24)$$

$-a < x_1, t < a$, Re [...] denotes the real part of the complex quantity inside the brackets[ ], and *const* is an arbitrary constant. From D_1 (24), we can reach D_2 using (14). Closed-form solutions are obtained after performing the various integrations in (24) using (16) for *f*. These are given below in the next section.

A comparison with a known result will be mentioned: Bilby and Eshelby [6] have given the solution of their equation (83) namely (the notations here are theirs)

$$\sigma_{yy}^{A} + A\int_{-a}^{a}\frac{D_y(x')dx'}{x-x'} = 0$$

with $\sigma_{yy}^{A} = -\alpha x$ to be

$$D_y(x) = \frac{\alpha}{A\pi}\left((a^2-x^2)^{1/2} - \frac{a^2}{2(a^2-x^2)^{1/2}}\right) + \frac{n}{\pi(a^2-x^2)^{1/2}}$$

where A, α, and n are constants (see in page 136 in Bilby and Eshelby (1968)). The first of our equation (13) with $\beta = 0$ is identical in form with their integral equation. Our result (24) agrees with the solution for D_y above after the integration with *f(t)* (16) has been performed (see also below).

III-2. Physical quantities associated with the interfacial crack

III-2-1. Dislocation distributions; Relative displacements of the faces of the crack

Closed-form expressions have been obtained for the distribution functions D_1 and D_2 of the edge crack dislocations. These are

$$D_n(x_1) = \frac{ch(\pi\delta)}{2bC\sqrt{a^2-x_1^2}}\,\mathrm{Re}\left[(i\delta_{n1}+\delta_{n2})\left\{2x_1(\sigma_{22}^{a} - 2a\delta\overline{\alpha})\right.\right.$$
$$\left.\left. - i\left(-4a\delta\sigma_{22}^{a} + \left(a^2(1+4\delta^2) - 2x_1^2\right)\overline{\alpha}\right)\right\}\left(\frac{a-x_1}{a+x_1}\right)^{i\delta}\right], \qquad (25)$$

$n = 1$ and 2. D_1 and D_2 only differ by a factor i in front of the curly brackets $\{\ \}$. ch is the hyperbolic cosine function.

The relative displacement ϕ_n ($n = 1$ and 2) of the faces of the crack in the x_n −direction is

$$\phi_n(x_1) = \int_{x_1}^{a} bD_n(x_1')dx_1' \,, \quad |x_1| \le a \,, \qquad (26)$$

and after integration

$$\phi_n(x_1) = \frac{ch(\pi\delta)}{C}\,\mathrm{Re}\left[(i\delta_{n1}+\delta_{n2})\frac{(a-x_1)^{i\delta+1/2}}{(2a)^{i\delta-1/2}}\left\{\sigma_{22}^{a}\left(F(X)-\frac{a-x_1}{2a(i\delta+3/2)}F(Y)\right)\right.\right.$$

$$\left.\left.+2ia\overline{\alpha}\left(\frac{1+2i\delta}{4}F(X)-\frac{a-x_1}{2a(i\delta+3/2)}\big(i\delta F(Y)+F(Z)\big)\right)\right\}\right], \quad (27)$$

$n=1$ and 2; F is Gauss's hypergeometric function and X, Y and Z are arguments for F with values

$$X=\big(i\delta+1/2, i\delta+1/2; i\delta+3/2; (a-x_1)/2a\big)$$
$$Y=\big(i\delta+1/2, i\delta+3/2; i\delta+5/2; (a-x_1)/2a\big)$$
$$Z=\big(i\delta-1/2, i\delta+3/2; i\delta+5/2; (a-x_1)/2a\big).$$

Other relations of interest (in the calculation of the crack extension force below, for instance) are expressions for ϕ_n, in the vicinity of the crack tip at $x_1=a$. At the distance $s=a-x_1$, $0<s<<a$, we get using (27)

$$\phi_n(s)=\frac{ch(\pi\delta)\sqrt{2as}}{C}\,\mathrm{Re}\left[(i\delta_{n1}+\delta_{n2})\left(\sigma_{22}^{a}-a\delta\overline{\alpha}+ia\overline{\alpha}/2\right)\left(\frac{s}{2a}\right)^{i\delta}\right] \quad (28)$$

$n=1$ and 2.

III-2-2. Stresses at the crack tip and crack extension force

We would like to express the total stresses $\overline{\sigma}_{12}$ and $\overline{\sigma}_{22}$ in the plane of the crack in the neighbourhood of the crack tip at $x_1=a$. These are given by the dominant terms of $(\overline{\sigma}_{ij}^{(1)}+\overline{\sigma}_{ij}^{(2)})$ in (6) at point $P=(x_1, x_2=0, x_3)$ with $x_1=a+s_1$, $0<s_1<<a$. Using (6 to 9) and (25) we obtain

$$\overline{\sigma}_{n2}(s_1)=\sqrt{\frac{a}{2s_1}}\,\mathrm{Re}\left[(i\delta_{n1}+\delta_{n2})(1+2i\delta)\left(\sigma_{22}^{a}-a\delta\overline{\alpha}+ia\overline{\alpha}/2\right)\left(\frac{s_1}{2a}\right)^{i\delta}\right], \quad (29)$$

$n=1$ and 2.

A procedure to calculate the crack extension force G has been described by Bilby and Eshelby [6]. We have also referred to it in a number of works (see Anongba [1, 8 to 11]).

This gives

$$G = \lim_{\Delta a \to 0} \frac{1}{\Delta a}\left(\frac{1}{2}\int_{a}^{a+\Delta a} \overline{\sigma}_{12}(s_1)\phi_1(s_2)dx_1 + \frac{1}{2}\int_{a}^{a+\Delta a} \overline{\sigma}_{22}(s_1)\phi_2(s_2)dx_1 \right); \quad (30)$$

for $\overline{\sigma}_{12}$ and $\overline{\sigma}_{22}$, we use (29) with $s_1 = x_1 - a$, and for ϕ_1 and ϕ_2 (28) with $s_2 = a + \Delta a - x_1$. We obtain

$$G = \frac{a\pi(1+4\delta^2)}{4C}\left(\left(\sigma_{22}^a - a\delta\overline{\alpha}\right)^2 + \left(\frac{a\overline{\alpha}}{2}\right)^2 \right). \quad (31)$$

When G is defined as

$$G = G_1 + G_2 \quad (32)$$

with

$$G_n = \lim_{\Delta a \to 0} \frac{1}{\Delta a}\left(\frac{1}{2}\int_{a}^{a+\Delta a} \overline{\sigma}_{n2}(s_1)\phi_n(s_2)dx_1 \right), \ n = 1 \text{ and } 2, \quad (33)$$

it follows

$$G_n = \frac{G}{2} + \left(-\delta_{n1} + \delta_{n2}\right)\frac{ach(\pi\delta)}{4C}$$
$$\times \lim_{\Delta a \to 0} \mathrm{Re}\left[(1+2i\delta)\eta^2 B\left(i\delta + 3/2, i\delta + 1/2\right)\left(\frac{\Delta a}{2a}\right)^{2i\delta} \right], \ n = 1 \text{ and } 2, \quad (34)$$

where

$$\eta = \sigma_{22}^a - a\delta\overline{\alpha} + ia\overline{\alpha}/2; \quad (35)$$

$B(x, y)$ is the beta function (Euler's integral of the first kind). It thus appears that G (31) is well defined, but G_n, associated with individual loading modes I and II, carries an oscillatory term with Δa indefinitely (see further comments below).

IV - DISCUSSION

When a material is loaded in a given direction, an internal stress develops in the corresponding perpendicular plane. This intrinsic phenomenon, common to most materials, is known as the Poisson's effect. In composites, additional shear stresses are induced across the interfaces. The present work is a first attempt to include Poisson's effect formally in the mathematical analysis of the conditions for crack propagation. This has been made possible by using a simple crack geometry, namely the interface crack of finite length 2*a*, loaded in tension in an infinitely extended surrounding medium. The framework is continuum linear elasticity. The study suggests that the induced shear stress (4) depends linearly on position along the interface. Physical quantities associated with the interface, ν_s (3) and μ_s and a_s (4), are evidenced. ν_s is the ratio of ϕ (distance along x_1 between A and B ***(Figure 1)*** in the welded state under load) by d (the corresponding distance in the non-welded state); we have

$$\nu_s = \frac{\phi}{d} = 1 + \frac{u_1(x_1) - u_2(x_1)}{d}. \qquad (36)$$

A maximum value for ν_s is one ($u_1 - u_2 = 0$) and a minimum is zero ($u_1 - u_2 = -d$). We have assumed that ν_s is constant in our analysis. By a direct inspection of ***Figure 1***, it may be seen that this is certainly correct over a relatively large x_1-interval. We shall use below the relation (31) for G with the aim of estimating the magnitude of the Poisson's effect; it is clear that taking $\nu_s = 1/2$ will be sufficient for this purpose. The other quantities μ_s and a_s (4) are the shear modulus about the interface and the separation distance along x_2 between two points A and B, located on either side of the interface, that suffered the action of the shear stress (4).

The magnitude of a_s must be sufficiently small but presumably not down to the atomic level; it would be better on the micron scale. Before we estimate the magnitude of Poisson's effect, it is interesting to compare our findings with previous results on the interface crack under load. A first advantage of the present work is that it gives various quantities, associated with the loaded crack, in closed-form expressions. This has been made possible with the help of the solution (24) to a class of singular integral equations (15). With D_1 and D_2 (25) on the one hand, and relative displacements ϕ_1 and ϕ_2 of the faces of the crack (27) on the other hand, a complete knowledge of the whole shape of the crack under load is achieved from $x_1 = -a$ to a. Our results agree with those obtained when Poisson's effect is neglected (i.e. $\overline{\alpha}$ (4) equal to zero). We can mention that G (31) with $\overline{\alpha} = 0$ is identical to the corresponding one displayed by Hutchinson [19] (see his relations (5) and (6)). Our analysis also reveals already known aspects on the crack extension force:

- G (31) the total energy release rate is well defined,
- G_1 and G_2 (33) the individual energy release rates associated with modes II and I carry an oscillatory term with δa indefinitely,
- If the oscillatory term is neglected, then $G_1 = G_2 = G/2$ (34).

The above behaviours for the crack extension force have been described earlier by Sun and Jih [20] and Raju et al. [21].
An estimate of the magnitude of the Poisson's effect in the conditions for the propagation of the interfacial crack loaded in tension is now under way in what follows. We define $\Delta\tilde{G} = \left(G(\overline{\alpha}) - G(\overline{\alpha} = 0)\right)/G(\overline{\alpha} = 0)$ where $G(\overline{\alpha})$ is the value of G (31) when $\overline{\alpha} \neq 0$; $G(\overline{\alpha} = 0)$, the value of G when $\overline{\alpha} = 0$, neglects Poisson's effect. We have

$$\Delta\tilde{G} = \kappa_s \left\{ \frac{(1+4\delta^2)\kappa_s}{4} \left(\frac{a}{a_s} \right)^2 - 2\delta \left(\frac{a}{a_s} \right) \right\} \tag{37}$$

where

$$\kappa_s = \nu_s \mu_s \left(\frac{\nu_1}{E_1} - \frac{\nu_2}{E_2} \right). \tag{38}$$

We further define $\tilde{a} = a/a_s$. $\Delta\tilde{G}$ is minimum for $\tilde{a} = \tilde{a}_e$ given by $\partial\Delta\tilde{G}(\tilde{a}_e)/\partial\tilde{a} = 0$. This gives

$$\tilde{a}_e = \frac{4\delta}{\kappa_s(1+4\delta^2)} \tag{39}$$

and

$$\Delta\tilde{G}(\tilde{a}_e) = -\frac{4\delta^2}{1+4\delta^2}. \tag{40}$$

Also, $\Delta\tilde{G}$ is zero for either $\tilde{a} = \tilde{a}_1 = 0$ or $\tilde{a} = \tilde{a}_2$ with

$$\tilde{a}_2 = \frac{8\delta}{\kappa_s(1+4\delta^2)}. \tag{41}$$

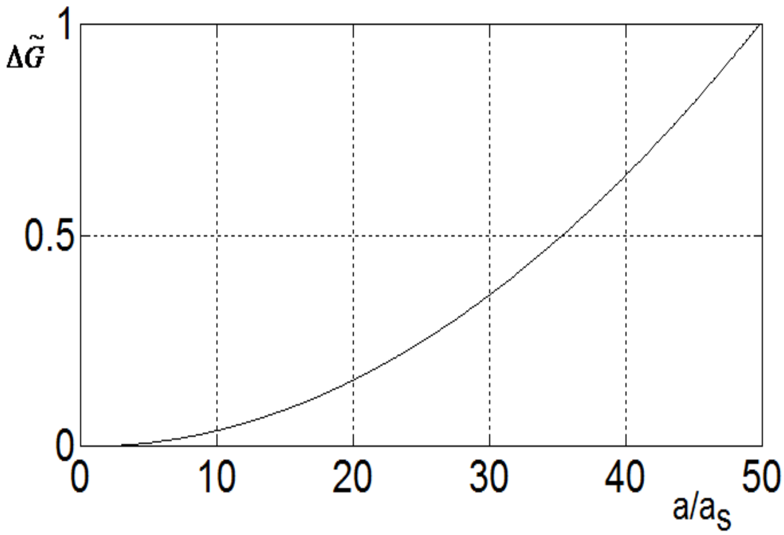

Figure 2 : *Positive relative values* $\Delta\tilde{G}$ *(37) of the reduced crack extension force* $\tilde{G} \equiv G(\overline{\alpha})/G(\overline{\alpha}=0)$ *as a function of crack half length a (normalized by* a_s*). Elastic constants of the system (1)/(2) correspond to MgO/Ni (see text).*

$\Delta\tilde{G}$ is plotted against $\tilde{a}$ in ***Figure 2*** for the system (1)/(2) corresponding to MgO/Ni with associated parameters taken from Hutchinson et al. [22]: $\delta = -0.0049$; $\mu_1 = 1.283\times10^{11}\,N/m^2$, $\nu_1 = 0.175$; $\mu_2 = 0.808\times10^{11}\,N/m^2$, $\nu_2 = 0.314$. The shear modulus is for polycrystalline materials. We also use $E_i = 2\mu_i(1+\nu_i)$ and take $\mu_s \cong \mu_2 = 0.9\times10^{11}\,N/m^2$, $\nu_s = 0.5$. $\Delta\tilde{G}$ ***(Figure 2)*** decreases from zero ($\tilde{a} = 0$) to a negative minimum $\Delta\tilde{G}(\tilde{a}_e) = -9.6\times10^{-5}$ ($\tilde{a}_e = 0.48$) and then increases indefinitely with $\tilde{a}$ taking again a value $\Delta\tilde{G}(\tilde{a}_2) = 0$ for $\tilde{a}_2 = 0.97$. Above $\tilde{a} = \tilde{a}_2$, $\Delta\tilde{G}$ takes a value of about one at $\tilde{a} = 50$. If an increase in the magnitude of the crack extension force is to be interpreted as a promotion of crack growth, ***Figure 2*** suggests that Poisson effect acts against interfacial crack extension for very small crack length but acts in favour of crack propagation for sufficiently large cracks.

We would like to stress that the way individual modes I and II (opening and sliding, respectively) are interconnected is intriguing. Although the externally applied loading σ_{22}^{a} is in the x_2 – direction (opening mode I loading), the sliding mode II seems to contribute on the equal scale to the crack extension force (see (31) to (34)). We know from studies on crack propagation in homogeneous materials that the crack kinks (out of the initial crack Ox_1x_3 – plane) when the sliding mode II contribution to G increases (for references, see in Anongba [11]). What is the trend for the interfacial crack? Much less is known. Experimental measurement of the stress to fracture an interface crack of length 2a, at the centre of the interface, as a function of crack half length (same geometry as in ***Figure 1***) is highly desirable to check basic modelling, in addition to interfacial surface energy measurements. In conclusion, because $\Delta\tilde{G}$ may be appreciably large, Poisson's effect is important for the conditions of the propagation of the interface crack loaded in tension.

V - CONCLUSION

The present study draws attention on the existence of an internal shear stress that originates from a difference in Poisson's contractions present in two different solids (1) and (2) firmly welded along a planar interface loaded in tension. The objective is to evaluate the contribution of this special internal shear stress to the conditions for the propagation of a brittle interfacial crack, of finite length, located at the centre of the interface. The framework is linear elasticity coupled with brittle fracture mechanics where the crack is represented by a continuous distribution of dislocations with infinitesimal Burgers vectors. The following points are brought into conclusion:

- The interfacial Poisson internal shear stress is assumed to be proportional to position along the interface.
- An analytical solution is given to a class of integral equations with Cauchy-type singular kernel involving the distribution functions of the interfacial crack dislocations.
- Closed-form expressions are given to the dislocation distribution functions, crack-tip stress and crack extension force.
- The crack extension force increases with crack length, for sufficiently large cracks.

In closure, detailed studies (both experimental and theoretical) on the contribution of the internal shear stress (promoted by Poisson's effect) to the conditions for the propagation of a crack, on or crossing the interface in composites, reveal all their importance.

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